COLLOQUIUM MATHEMATICUM

Online First version

EXCEPTIONAL MODULES OVER WILD CANONICAL ALGEBRAS

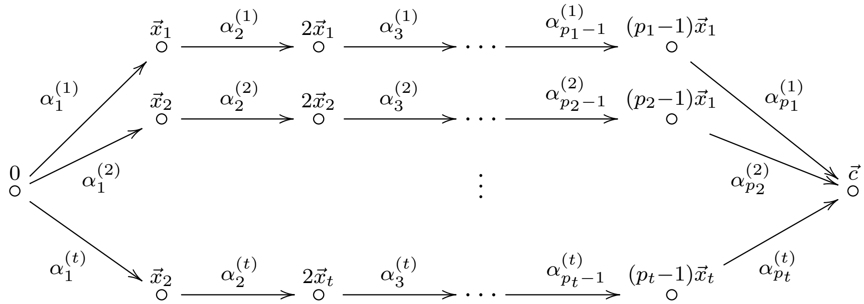
BY

DAWID EDMUND KĘDZIERSKI and HAGEN MELTZER (Szczecin)

Abstract. We show that in a certain sense "almost all" exceptional modules over wild canonical algebra $\Lambda(\underline{p}, \underline{\lambda})$ can be described by matrices with entries of the form $\lambda_i - \lambda_j$, where λ_i, λ_j are elements from the parameter sequence $\underline{\lambda}$.

The proof is based on Schofield induction for sheaves in the associated categories of weighted projective lines (Kędzierski and Meltzer 2013) and an extended version of Ringel's proof of the 0, 1 matrix property of exceptional representations for finite acyclic quivers.

1. Introduction. Canonical algebras were introduced by C. M. Ringel [Rin84]. A *canonical algebra* Λ of quiver type over a field k is the quotient algebra of the path algebra of a quiver Q:



modulo an ideal I defined by the *canonical relations*

$$\alpha_{p_i}^{(i)} \ldots \alpha_2^{(i)} \alpha_1^{(i)} = \alpha_{p_1}^{(1)} \ldots \alpha_2^{(1)} \alpha_1^{(1)} + \lambda_i \alpha_{p_2}^{(2)} \ldots \alpha_2^{(2)} \alpha_1^{(2)} \quad \text{for } i = 3, \ldots, t,$$

where the λ_i are pairwise distinct non-zero elements of k, called *parameters*. The positive integers p_i are at least 2 and are called *weights*. Usually we assume that k is algebraically closed, but for many results this is not necessary. The algebra Λ depends on the weight sequence $\underline{p} = (p_1, \ldots, p_t)$ and

2020 *Mathematics Subject Classification*: 16G20, 14F05, 16G60.

Key words and phrases: exceptional module, canonical algebra, wild type, zero-one matrix, Schofield induction, weighted projective line, exceptional pair, generalized Kronecker algebra.

Received 25 March 2019; revised 15 July 2019.

Published online *.

DOI: 10.4064/cm7876-8-2019

the sequence of parameters $\underline{\lambda} = (\lambda_2, \ldots, \lambda_t)$. We can assume that $\lambda_2 = 0$ and $\lambda_3 = 1$. We write $\Lambda = \Lambda(\underline{p}, \underline{\lambda})$. Concerning the complexity of the module category over Λ there are three types of canonical algebras: domestic, tubular and wild (see [ASS06] for definitions). It was shown in [Rin76] that the canonical algebra Λ is of wild type if and only if the Euler characteristic $\chi_\Lambda = (2 - t) + \sum_{i=1}^{t} 1/p_i$ is negative.

Let $\Lambda = \Lambda(\underline{p}, \underline{\lambda})$ be a canonical algebra and let $Q = (Q_0, Q_1)$ be its quiver, where Q_0 is the set of vertices and Q_1 the set of arrows. Denote by $\mathrm{mod}(\Lambda)$ the category of finitely generated right Λ-modules. Then a right module M in $\mathrm{mod}(\Lambda)$ can be viewed as a k-linear representation $M = (M_j, M_\beta)$, where M_j is a finite-dimensional k-vector space for each $j \in Q_0$, and $M_\beta : M_i \to M_j$ is a k-linear map for any arrow $\beta : j \to i$, such that the canonical relations are satisfied. We will usually identify linear maps with matrices.

We recall that a right module M over an algebra A is defined to be *exceptional* if its endomorphism algebra $\mathrm{End}_A(M)$ is a skew field and M does not have self-extensions, i.e. $\mathrm{Ext}^i_A(M, M) = 0$ for $i > 0$. For a canonical algebra Λ the second condition can be reduced to $\mathrm{Ext}^1_\Lambda(M, M) = 0$. Moreover, since k is algebraically closed, the first condition implies that $\mathrm{End}_\Lambda(M) = k$.

Our aim is to study the possible entries of the matrices of the k-linear maps $M_\beta : M_i \to M_j$ of an exceptional module M over a wild canonical algebra Λ.

The issue of possible coefficients that appear in the matrices describing exceptional modules over various algebras, or more generally exceptional objects in some category, has been considered many times before. In 1998 C. M. Ringel [Rin88] proved that each exceptional representation of a finite acyclic quiver can be realized by $0, 1$ matrices. The same result was shown by P. Dräxler [Drä01] for indecomposable modules over representation-finite algebras. In the case of the path algebra of a Dynkin quiver, P. Gabriel [Gab71] computed $0, 1$ matrices for all indecomposable representations. A similar result was shown by M. Kleiner [Kle72] for indecomposable representations of some representation-finite posets, and a complete list is presented by D. Simson [Sim91] and by Arnold–Richman [AHR92] (see also a result of K. J. Bäckström [Bäc72] for orders over lattices). For the path algebra of an extended Dynkin quiver indecomposable representations allow $0, \pm 1$ matrices (see [KuM07b], [KęM11]). Among new results we mention a paper of M. Grzecza, S. Kasjan and A. Mróz [GKM12].

The problem of possible entries in matrices of exceptional representations was well researched in the case of canonical algebras of domestic and tubular type. For domestic type, if $\mathrm{char}\, k \neq 2$, D. Kussin and the second author [KuM07a] computed matrices having 0, ± 1 entries for all indecomposable modules, where -1 entries appear only for very special regular modules. The case of arbitrary characteristic was discussed in [KoM08].

A tubular canonical algebra has one of the following weight types: $(2,3,6)$, $(3,3,3)$, $(2,4,4)$ and $(2,2,2,2)$ with a parameter λ. In the first three cases each exceptional module can be described by matrices having 0, ± 1 entries. In the last case entries 0, ± 1, $\pm\lambda$ and $1-\lambda$ can appear. This result was discussed in [Mel07] and the proof uses universal extensions in the sense of K. Bongartz [Bon81].

Later P. Dowbor, A. Mróz and the second author [DMM10] developed an algorithm and a computer program for explicit calculations of matrices for exceptional modules over tubular canonical algebras.

In general little is known about matrices of non-exceptional modules. However, in the case of tubular canonical algebras an algorithm for the computation of matrices of non-exceptional modules was developed in [DMM14b]. Moreover, explicit formulas for these matrices were obtained when the module is of integral slope [DMM14a].

Recently the $0,1$ property was proved for exceptional objects in the category of nilpotent operators of vector spaces with one invariant subspace, where the nilpotency degree is bounded by 6 [DMS19] and for exceptional objects in the category of nilpotent operators of vector spaces with two incomparable invariant subspaces, where the nilpotency degree is bounded by 3 [DM19]. Both problems are of tubular type and are related to recent results on stable vector space categories [KLM13a], [KLM13b], [KLM18], and the Birkhoff problem [Bir34], which was studied recently by M. Schmidmeier and C. M. Ringel [RSc08] and D. Simson [Sim18].

The aim of this paper is to present the following result.

Main Theorem 1.1. *Let $\Lambda = \Lambda(\underline{p}, \underline{\lambda})$ be a wild canonical algebra of quiver type with $\underline{\lambda} = (\lambda_2, \ldots, \lambda_t)$. Then "almost all" exceptional Λ-modules can be realized by matrices with entries of the form $\lambda_i - \lambda_j$, where $2 \leq i, j \leq t$.*

"Almost all" means that in every $\tau_{\mathbb{X}}$-orbit of exceptional modules, to the right of a certain place, all modules have the expected matrices. We strongly believe that the theorem holds for all exceptional Λ-modules, but the proof would need additional arguments.

The theorem will be proved by induction on the rank of a module. Recall that matrices for modules of rank 0 and 1 are known [KuM07a], [Mel07]. Next, by Schofield induction [Sch91] each exceptional Λ-module M of rank greater than or equal to 2 can be obtained as the central term of a non-split sequence

$$(\star) \qquad 0 \to Y^{\oplus v} \to M \to X^{\oplus u} \to 0,$$

where (X, Y) is an orthogonal exceptional pair in the category $\mathrm{coh}(\mathbb{X})$ of coherent sheaves over the weighted projective line $\mathbb{X}$ corresponding to Λ, and (u, v) is the dimension vector of an exceptional representation for the generalized Kronecker algebra having $\dim_k \mathrm{Ext}^1_{\mathbb{X}}(X, Y)$ arrows [KęM13]. Consequently, following C. M. Ringel [Rin88] we will study the category $\mathcal{F}(X,Y)$ of all middle terms of short exact sequences $(\star)$ for $u,v \in \mathbb{N}_0$. This category is equivalent to the module category of the generalized Kronecker algebra. Finally, using an alternative description of extension spaces, by using Ringel's map [Rin88] we will determine the relevant matrix entries for exceptional modules over wild canonical algebras.

Throughout this paper we freely use the quiver representation terminology of [ASS06]. For the representation type terminology and known facts on tameness wildness, domesticity, tubularity, and related subjects the reader is referred to [Rin84] and [SSk06].

2. Notations and basic concepts. We recall the concept of a weighted projective line in the sense of Geigle–Lenzing [GL87] associated to a canonical algebra $\Lambda = \Lambda(\underline{p},\underline{\lambda})$. Let $\mathbb{L} = \mathbb{L}(\underline{p})$ be the rank 1 abelian group with generators $\vec{x}_1,\ldots,\vec{x}_t$ and relations $p_1\vec{x}_1 = \cdots = p_t\vec{x}_t := \vec{c}$, where $\vec{c}$ is called the *canonical element*. Moreover $\mathbb{L}$ is an ordered group with $\mathbb{L}_+ = \sum_{i=1}^{t} \mathbb{N}_0\vec{x}_i$ as its set of non-negative elements. Each element $\vec{y}$ of $\mathbb{L}$ can be written in *normal form* $\vec{y} = a\vec{c} + \sum_{i=1}^{t} a_i\vec{x}_i$ with $a \in \mathbb{Z}$ and $0 \leq a_i < p_i$ then $\vec{y} \geq 0$ when $a \geq 0$.

The polynomial algebra $k[x_1,\ldots,x_t]$ is $\mathbb{L}$-graded, where the degree of x_i is $\vec{x}_i$. Because the polynomials $f_i = x_i^{p_i} - x_1^{p_1} - \lambda_i x_2^{p_2}$ for $i = 3,\ldots,t$ are homogeneous, the quotient algebra $S = k[x_1,\ldots,x_t]/\langle f_i \mid i = 3,\ldots,t\rangle$ is also $\mathbb{L}$-graded. A *weighted projective line* $\mathbb{X}$ is by definition the projective spectrum of the $\mathbb{L}$-graded algebra S. The category of coherent sheaves over $\mathbb{X}$ will be denoted by $\mathrm{coh}(\mathbb{X})$. In other words, $\mathrm{coh}(\mathbb{X})$ is the Serre quotient $\mathrm{mod}^{\mathbb{L}}(S)/\mathrm{mod}_0^{\mathbb{L}}(S)$, where $\mathrm{mod}^{\mathbb{L}}(S)$ is the category of finitely generated $\mathbb{L}$-graded modules over S, and $\mathrm{mod}_0^{\mathbb{L}}(S)$ the subcategory of modules of finite length. It is well known that each indecomposable sheaf in $\mathrm{coh}(\mathbb{X})$ is either a locally free sheaf, called a *vector bundle*, or a *sheaf of finite length*. Denote by $\mathrm{vect}(\mathbb{X})$ (resp. $\mathrm{coh}_0(\mathbb{X})$) the category of vector bundles (resp. finite length sheaves) on $\mathbb{X}$.

Recall from [GL87] that for coherent sheaves there are well known invariants, *rank*, *degree* and *determinant*, which correspond to linear forms $\mathrm{rk}, \deg : K_0(\mathbb{X}) \to \mathbb{Z}$ and $\det : K_0(\mathbb{X}) \to \mathbb{L}(\underline{p})$, again called *rank*, *degree* and *determinant*, where $K_0(\mathbb{X})$ is the Grothendieck group of $\mathrm{coh}(\mathbb{X})$.

A vector bundle of rank 1 is called a *line bundle*. Each line bundle has the form $\mathcal{O}(\vec{x})$ for some $\vec{x} \in \mathbb{L}$, where $\mathcal{O}$ is the structure sheaf. Moreover there is an isomorphism $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}),\mathcal{O}(\vec{y})) \cong S_{\vec{y}-\vec{x}}$, where $S_{\vec{y}-\vec{x}}$ is the space of homogeneous elements of the algebra S of degree $\vec{y}-\vec{x}$, and $S_{\vec{y}-\vec{x}} \neq 0$ if and only if $\vec{y}-\vec{x} \geq 0$.

The category $\mathrm{coh}(\mathbb{X})$ is a Hom-finite, abelian k-category. Moreover, it is hereditary in the sense that $\mathrm{Ext}^i_{\mathbb{X}}(-,-) = 0$ for $i \geq 2$, and it has Serre duality in the form $\mathrm{Ext}^1_{\mathbb{X}}(F,G) \cong D\,\mathrm{Hom}_{\mathbb{X}}(G,\tau_{\mathbb{X}}F)$, where the Auslander–

Reiten translation $\tau_{\mathbb{X}}$ is given by the shift $F \mapsto F(\vec{\omega})$, where $\vec{\omega} := (t-2)\vec{c} - \sum_{i=1}^{t} \vec{x}_i$ denotes the *dualizing element*; equivalently, the category $\mathrm{coh}(\mathbb{X})$ has Auslander–Reiten sequences. Moreover, there is a tilting object composed of line bundles with $\mathrm{End}_{\mathbb{X}}(T) = \Lambda$ and it induces an equivalence of bounded derived categories $\mathcal{D}^b(\mathrm{coh}(\mathbb{X})) \overset{\cong}{\to} \mathcal{D}^b(\mathrm{mod}(\Lambda))$.

For more details concerning the concept of weighted projective lines and the category of coherent sheaves over them we recommend [GL87] and [LeM93].

Recall that a coherent sheaf E over $\mathbb{X}$ is called *exceptional* if $\mathrm{Ext}^1_{\mathbb{X}}(E,E) = 0$ and $\mathrm{End}_{\mathbb{X}}(E)$ is a division ring; in case k is algebraically closed, the latter means that $\mathrm{End}_{\mathbb{X}}(E) = k$. A pair (X, Y) in $\mathrm{coh}(\mathbb{X})$ is called *exceptional* if X and Y are exceptional and $\mathrm{Hom}_{\mathbb{X}}(Y,X) = 0 = \mathrm{Ext}^1_{\mathbb{X}}(Y,X)$. Finally, an exceptional pair is *orthogonal* if additionally $\mathrm{Hom}_{\mathbb{X}}(X,Y) = 0$.

The *rank* of a Λ-module is defined to be $\mathrm{rk}\, M := \dim_k M_0 - \dim_k M_{\vec{c}}$. The rank of a module in this sense equals the rank of the corresponding sheaf in the geometric sense. We denote by $\mathrm{mod}_+(\Lambda)$ (respectively $\mathrm{mod}_-(\Lambda)$ or $\mathrm{mod}_0(\Lambda)$) the full subcategory of all Λ-modules whose indecomposable summands all have positive (respectively negative or zero) rank. Similarly, by $\mathrm{coh}_+(\mathbb{X})$ (resp. $\mathrm{coh}_-(\mathbb{X})$) we denote the full subcategory of all vector bundles over $\mathbb{X}$ such that the functor $\mathrm{Ext}^1_{\mathbb{X}}(T,-)$ (resp. $\mathrm{Hom}_{\mathbb{X}}(T,-)$) vanishes. Under the equivalence $\mathcal{D}^b(\mathrm{coh}(\mathbb{X})) \overset{\cong}{\to} \mathcal{D}^b(\mathrm{mod}(\Lambda))$,

- $\mathrm{coh}_+(\mathbb{X})$ corresponds to $\mathrm{mod}_+(\Lambda)$ by means of $E \mapsto \mathrm{Hom}_{\mathbb{X}}(T,E)$,
- $\mathrm{coh}_0(\mathbb{X})$ corresponds to $\mathrm{mod}_0(\Lambda)$ by means of $E \mapsto \mathrm{Hom}_{\mathbb{X}}(T,E)$,
- $\mathrm{coh}_-(\mathbb{X})[1]$ corresponds to $\mathrm{mod}_-(\Lambda)$ by means of $E[1] \mapsto \mathrm{Ext}^1_{\mathbb{X}}(T,E)$, where $[1]$ denotes the suspension functor of the triangulated category $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$.

For simplicity we will often identify a sheaf E in $\mathrm{coh}_+(\mathbb{X})$ or $\mathrm{coh}_0(\mathbb{X})$ with the corresponding Λ-module $\mathrm{Hom}_{\mathbb{X}}(T,E)$.

3. Exceptional modules of small rank. First, we start with some matrix notations. For a natural number n we denote by I_n the $n \times n$ identity matrix we consider. For natural numbers n and m we consider the matrices

$$X_{n+m,n} := \left[\begin{array}{ccc} & & \\ & I_n & \\ & & \\ \hline 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{array}\right], \qquad Y_{n+m,n} := \left[\begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ \hline & & \\ & I_n & \\ & & \end{array}\right]$$

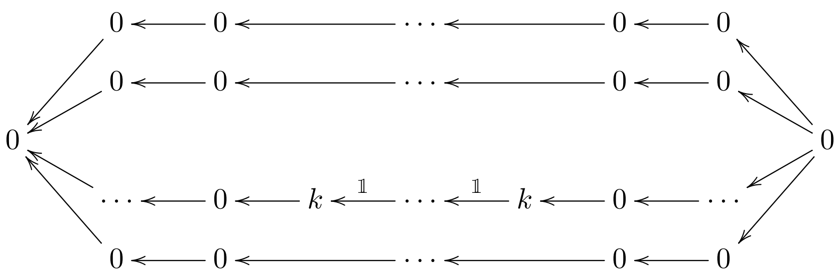
in $M_{n+m,n}(k)$.

A Λ-module of rank zero is called *regular*. It is well known (see [LdP97]) that the Auslander–Reiten quiver of all regular Λ-modules consists of a family of orthogonal regular tubes with t exceptional tubes $\mathcal{T}_1, \dots, \mathcal{T}_t$ of rank $p_1, \dots, p_t$, respectively, while the other tubes are homogeneous. Moreover an exceptional regular module lies in an exceptional tube and its quasi-length is less than the rank of the tube [Rin84]. We will use the description of indecomposable regular modules from [KuM07a]. However, we will only quote the shape of the exceptional ones, which lie in the tube $\mathcal{T}_i$ for $i \in \{3, \dots, t\}$. For $\mathcal{T}_1$ and $\mathcal{T}_2$ the description is similar. Following the notations introduced in [KuM07a] we denote by $S_a^{[l]}$ a regular Λ-module, with quasi-length l, where a indicates the position on the corresponding floor of the tube. For an exceptional module $S_a^{[l]}$ we have $l < p_i$ and so all vector spaces of $S_a^{[l]}$ are zero- or one-dimensional.

There are three cases:

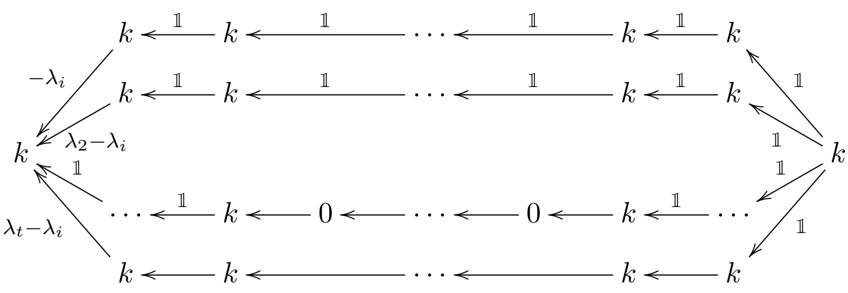
(1) $1 \le a < p_i$ and $0 < l < p_i - a$,
(2) $1 \le a < p_i$ and $p_i - a < l < p_i$,
(3) $a = p_i$ and $0 < l < p_i$.

Case (1). Then $S_a^{[l]}$ has the form



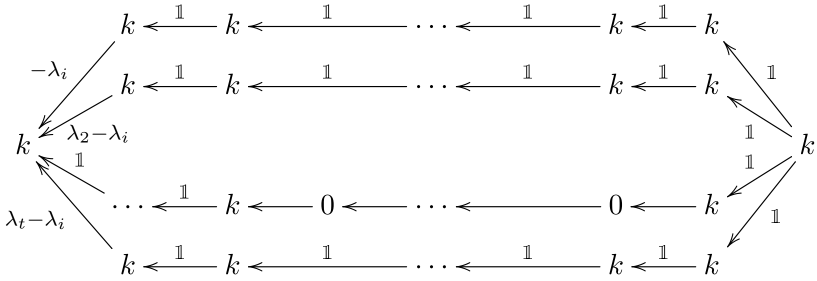
where $0 \leftarrow k$ and $k \leftarrow 0$ correspond to the arrows $\alpha_a^{(i)}$ and $\alpha_{a+l}^{(i)}$ in the ith arm.

Case (2). Let $s := l - (p_i - a)$. Then $S_a^{[l]}$ has the form



where $k \leftarrow 0$ and $0 \leftarrow k$ correspond to the arrows $\alpha_s^{(i)}$ and $\alpha_a^{(i)}$ in the ith arm.

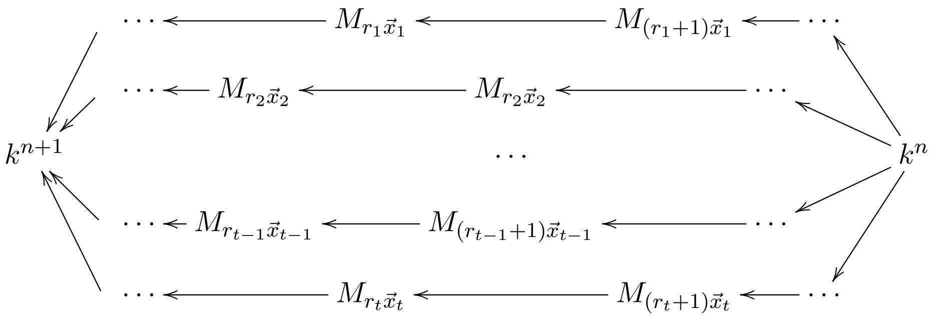
Case (3). Then $S_a^{[l]}$ has the form



where $k \leftarrow 0$ and $0 \leftarrow k$ correspond to the arrows $\alpha_l^{(i)}$ and $\alpha_{p_i}^{(i)}$ in the ith arm.

For Λ-modules of rank 1 we have the following description.

Proposition 3.1 ([Mel07]). *Let Λ be a canonical algebra of quiver type and of arbitrary representation type and M an exceptional Λ-module of rank 1. Then M is isomorphic to one of the following modules:*



where r_i is an integer such that $0 \le r_i < p_i$ for each $i = 1, \dots, t$ and

$$M_{s\vec{x}_i} = \begin{cases} k^{n+1} & \text{for } 0 \le s \le r_i, \\ k^n & \text{for } r_i < s \le p_i. \end{cases}$$

Further the matrices of M are as follows:

$$M_{\alpha_s^{(i)}} = \begin{cases} I_{n+1} & \text{for } 1 < s < r_i \\ I_n & \text{for } r_i < s \le p_i \end{cases} \qquad \text{for } i = 1, \dots, t,$$

$$M_{\alpha_{r_1}^{(1)}} = X_{n+1,n},$$

$$M_{\alpha_{r_2}^{(2)}} = Y_{n+1,n},$$

and for $i = 3, \dots, t$ we distinguish two cases:

(a) $r_i = 0$:

$$M_{\alpha_1^{(i)}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & \lambda_i \end{bmatrix} \in M_{n+1,n}(k);$$

(b) $r_i > 0$:

$$M_{\alpha_1^{(i)}} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \lambda_i & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \lambda_i & 1 \end{bmatrix} \in M_{n+1}(k), \qquad M_{\alpha_{r_i}^{(i)}} = X_{n+1,n}.$$

4. Schofield induction from sheaves to modules. Let M be an exceptional object from $\mathrm{mod}_+(\Lambda)$ of rank greater than or equal to 2. Then there is a short exact sequence

$$(\star) \qquad 0 \to Y^{\oplus v} \to M \to X^{\oplus u} \to 0,$$

where (X, Y) is an orthogonal exceptional pair in the category $\mathrm{coh}(\mathbb{X})$ such that $\mathrm{rk}\, X < \mathrm{rk}\, M$ and $\mathrm{rk}\, Y < \mathrm{rk}\, M$, and (u, v) is the dimension vector of an exceptional representation of the generalized Kronecker algebra given by the quiver

$$\Theta(n): \quad 1 \underset{\alpha_n}{\overset{\alpha_1}{\underset{\vdots}{\rightrightarrows}}} 2$$

with $n := \dim_k \mathrm{Ext}^1_{\mathbb{X}}(X, Y)$ arrows. This result is called *Schofield induction* [Sch91] and was applied by C. M. Ringel in the situation of exceptional representations over finite acyclic quivers, hence of hereditary algebras [Rin88].

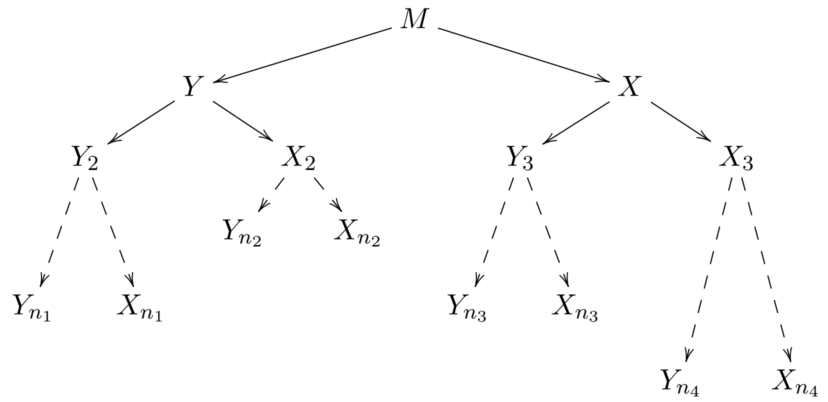
When the rank of X or Y is at least 2, we can apply Schofield induction again to obtain the sequences

$$0 \to Y_2^{\oplus v_2} \to Y \to X_2^{\oplus u_2} \to 0, \qquad 0 \to Y_3^{\oplus v_3} \to X \to X_3^{\oplus u_3} \to 0.$$

Because with each successive use of Schofield induction, the rank of the sheaves decreases, after a finite number of steps we get pairs of exceptional sheaves of rank 0 or 1.

This situation is illustrated by the following diagram, which has the shape of a tree:

(4.1)



Applying the functor $\mathrm{Ext}^1_{\mathbb{X}}(T,-)$ to the exact sequence $(\star)$ we see that if M is a Λ-module, then each sheaf X_{i_n} such that there is a path from M to X_{i_n} in the tree (4.1) is also a Λ-module. However, we do not know whether a sheaf Y_* is a Λ-module.

The following lemma will allow us, by using the $\tau_{\mathbb{X}}$-translation, to shift the tree (4.1) so that all its components will be Λ-modules.

Lemma 4.1. *Let $\{L_1,\dots,L_m\}$ be a family of line bundles over $\mathbb{X}$. Then there is a natural number N such that $\mathrm{Ext}^1_{\mathbb{X}}(T,\tau^n_{\mathbb{X}}L_j)=0$ for $j=1,\dots,m$ and for all $n>N$.*

Proof. Let $L_j=\mathcal{O}(a_j\vec{c}+\sum_{i=1}^t b_{j,i}\vec{x}_i)$, where $a_j\in\mathbb{Z}$, $0\le b_{j,i}\le p_i-1$ for $j=1,\dots,m$ and $i=1,\dots,t$. We put

$$N:=\max\{\lfloor(1-a_j)(t-2)\rfloor+1\mid 1\le j\le m\}.$$

Then

$$\vec{c}+\vec{\omega}-\det\tau^n_{\mathbb{X}}L_j=\big(1-a_j-(n-1)(t-2)\big)\vec{c}+\sum_{i=1}^t(n-1-b_{j,i})\vec{x}_i<0$$

for all $n>N$ and $j=1,\dots,m$. Therefore by Serre duality

$$(\triangle)\qquad \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}),\tau^n_{\mathbb{X}}L_j)\cong D\,\mathrm{Hom}_{\mathbb{X}}(\tau^n_{\mathbb{X}}L_j,\mathcal{O}(\vec{c}+\vec{\omega}))=0\quad \text{for } j=1,\dots,m.$$

We have to show that if $n>N$ then $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{x}),\tau^n_{\mathbb{X}}L_j)=0$ for $0\le\vec{x}<\vec{c}$ and $j=1,\dots,m$. Suppose that $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{x}),\tau^n_{\mathbb{X}}L_j)\neq 0$ for some $0\le\vec{x}<\vec{c}$. Then by Serre duality, $\det\tau^n_{\mathbb{X}}L_j\le\vec{x}+\vec{\omega}$. As $\vec{x}+\vec{\omega}\le\vec{c}+\vec{\omega}$, we get $\det\tau^n_{\mathbb{X}}L_j\le\vec{c}+\vec{\omega}$ and $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}),\tau^n_{\mathbb{X}}L_j)\cong D\,\mathrm{Hom}_{\mathbb{X}}(\tau^n_{\mathbb{X}}L_j,\mathcal{O}(\vec{c}+\vec{\omega}))\neq 0$ contrary to $(\triangle)$. ∎

Immediately from the lemma above we deduce the following corollary.

Corollary 4.2. *There is a natural number N such that for $n>N$ all components of the tree* (4.1) *shifted by $\tau^n_{\mathbb{X}}$ are Λ-modules.*

Proof. First, note that if the sheaves X and Y in the sequence $(\star)$ are Λ-modules, then the middle term M is a Λ-module. Next, since there are no non-zero morphisms from finite length sheaves to vector bundles, each finite length sheaf is a Λ-module.

Let $\mathcal{L} = \{L_i\}_{i\in I}$ be the set of all line bundles appearing in (4.1). From Lemma 4.1 applied to $\mathcal{L}$, there is a natural number N such that for all $n > N$ the line bundles $\tau_{\mathbb{X}}^n L_i$ are Λ-modules for $i \in I$. So the vector bundles in the penultimate level of the tree (4.1) are also Λ-modules. Moving up from the bottom, we infer that all sheaves in the $\tau_{\mathbb{X}}^n$ image are Λ-modules. ∎

5. Description of extension spaces. Let (X, Y) be an orthogonal exceptional pair in the category $\mathrm{coh}(\mathbb{X})$, so $\mathrm{Hom}_\Lambda(X,Y) = 0 = \mathrm{Hom}_\Lambda(Y,X)$, $\mathrm{Ext}^1_\Lambda(Y,X) = 0$ and $\mathrm{Ext}^1_\Lambda(X,Y) = k^n$ is a non-zero space. Assume further that both sheaves X and Y in the sequence $(\star)$ are Λ-modules.

We consider the category $\mathcal{F}(X,Y)$ of all right Λ-modules M that appear as the middle term in a short exact sequence

$$0 \to Y^{\oplus v} \to M \to X^{\oplus u} \to 0 \quad \text{for some } v, u \in \mathbb{N}_0.$$

It is well known (see [Rin76]) that the category $\mathcal{F}(X,Y)$ is abelian and has only two simple objects, X and Y, where the former is injective and the latter is projective.

Acting like C. M. Ringel [Rin88] in the situation of modules over a hereditary algebra we show that the problem of classifying the objects in $\mathcal{F}(X,Y)$ can be reduced to the classification of modules over the generalized Kronecker algebra with n arrows.

To do so let $\eta_1, \ldots, \eta_n$ be a basis of the vector space $\mathrm{Ext}^1_\Lambda(X,Y)$. Thus we have short exact sequences

$$\eta_i: \quad 0 \to Y \to Z_i \to X \to 0 \quad \text{for } i = 1, \ldots, n.$$

From the "pull-back" construction there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y^{\oplus n} & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle [1_X,\ldots,1_X]^T} & & \\
0 & \longrightarrow & Y^{\oplus n} & \longrightarrow & \bigoplus_{i=1}^n Z_i & \longrightarrow & X^{\oplus n} & \longrightarrow & 0
\end{array}$$

where the upper sequence is a universal extension and Z is an exceptional projective object in $\mathcal{F}(X,Y)$. In addition, the projective module $Y \oplus Z$ is a progenerator of $\mathcal{F}(X,Y)$. Therefore the functor $\mathrm{Hom}_{\mathbb{X}}(Y \oplus Z, -)$ induces an equivalence between $\mathcal{F}(X,Y)$ and the category of modules over the algebra $\mathrm{End}_\Lambda(Y \oplus Z)$, which is isomorphic to the generalized Kronecker algebra $k\Theta(n)$, where $n := \dim_k \mathrm{Ext}^1_\Lambda(X,Y)$.

Now, we need a more precise description of the above equivalence. Recall from [Rin88] the concept of extension space between two quiver representations X and Y. Let $C^0(X,Y)$ and $C^1(X,Y)$ be the vector spaces defined by

$$C^0(X,Y) := \bigoplus_{0\le\vec{x}\le\vec{c}} \mathrm{Hom}_k(X_{\vec{x}}, Y_{\vec{x}}), \qquad C^1(X,Y) := \bigoplus_{\alpha:\vec{x}\to\vec{y}} \mathrm{Hom}_k(X_{\vec{y}}, Y_{\vec{x}}),$$

and let $\delta_{X,Y} : C^0(X,Y) \to C^1(X,Y)$ be the linear map, defined by

$$\delta_{X,Y}([f_{\vec{x}}]_{0\le\vec{x}\le\vec{c}}) = [f_{\vec{y}}X_\alpha - Y_\alpha f_{\vec{z}}]_{\alpha:\vec{y}\to\vec{z}},$$

where α runs through Q_1.

For the path algebra kQ the map $\delta_{X,Y} : C^0(X,Y) \to C^1(X,Y)$ gives a useful description of the extension space of kQ-modules [Rin88]. Indeed, there is then a k-linear isomorphism

$$\mathrm{Ext}^1_{kQ}(X,Y) \cong C^1(X,Y)/\mathrm{Im}(\delta_{X,Y}).$$

For modules over a canonical algebra $\Lambda = \Lambda(\underline{p},\underline{\lambda})$ we must additionally consider the canonical relations of the algebra Λ. For this we take the subspace $U(X,Y)$ of $C^1(X,Y)$ consisting of all $[f_\alpha]_{\alpha\in Q_1}$ satisfying

$$\begin{aligned}
&Y_{\omega^{(i)}_{1,p_i-1}} f_{\alpha^{(i)}_{p_i}} + Y_{\omega^{(i)}_{1,p_i-2}} f_{\alpha^{(i)}_{p_i-1}} X_{\alpha^{(i)}_{p_i}} + \cdots + Y_{\alpha^{(i)}_1} f_{\alpha^{(i)}_2} X_{\omega^{(i)}_{3,p_i}} + f_{\alpha^{(i)}_1} X_{\omega^{(i)}_{2,p_i}} \\
&= Y_{\omega^{(1)}_{1,p_1-1}} f_{\alpha^{(1)}_{p_1}} + Y_{\omega^{(1)}_{1,p_1-2}} f_{\alpha^{(1)}_{p_1-1}} X_{\alpha^{(1)}_{p_1}} + \cdots + Y_{\alpha^{(1)}_1} f_{\alpha^{(1)}_2} X_{\omega^{(1)}_{3,p_1}} + f_{\alpha^{(1)}_1} X_{\omega^{(1)}_{2,p_1}} \\
&\quad + \lambda_i \big(Y_{\omega^{(2)}_{1,p_2-1}} f_{\alpha^{(2)}_{p_2}} + Y_{\omega^{(2)}_{1,p_2-2}} f_{\alpha^{(2)}_{p_2-1}} X_{\alpha^{(2)}_{p_2}} + \cdots + Y_{\alpha^{(2)}_1} f_{\alpha^{(2)}_2} X_{\omega^{(2)}_{3,p_2}} + f_{\alpha^{(2)}_1} X_{\omega^{(2)}_{2,p_2}} \big)
\end{aligned}$$

for $i = 3, \ldots, t$.

Lemma 5.1 ([Mel07]). $\mathrm{Ext}^1_\Lambda(X,Y) \cong U(X,Y)/\mathrm{Im}(\delta_{X,Y})$.

We recall the definition of the isomorphism above. Choosing bases of the spaces $M_{\vec{x}}$ we can assume that for each arrow $\alpha : i \to j$ the corresponding map $M_\alpha : M_j \to M_i$ has the shape

$$\left[\begin{array}{c|c} Y_\alpha & \varphi_\alpha \\ \hline 0 & X_\alpha \end{array}\right].$$

Then an isomorphism $\phi : \mathrm{Ext}^1_\Lambda(X,Y) \to U(X,Y)/\mathrm{Im}(\delta_{X,Y})$ is given by the formula $M = (M_i, M_\alpha) \overset{\phi}{\mapsto} (\varphi_\alpha)_{\alpha\in Q_0} + \mathrm{Im}(\delta_{X,Y})$.

Now, we can describe Λ-modules contained in $\mathcal{F}(X,Y)$ by using the matrices of X, Y and the representation of the quiver $\Theta(n)$, which corresponds to the module M. Each module in $\mathcal{F}(X,Y)$ can be identified with an element of $\mathrm{Ext}^1_\Lambda(X^{\oplus u}, Y^{\oplus v})$. Since $X^{\oplus u} = X \otimes k^u$ and $Y^{\oplus v} = Y \otimes k^v$, the space $\mathrm{Ext}^1_\Lambda(X^{\oplus u}, Y^{\oplus v}) = \mathrm{Ext}^1_\Lambda(X \otimes k^u, Y \otimes k^v)$ is given by the map $\delta_{X\otimes k^u, Y\otimes k^v}$, where the tensor product is taken over k. In this situation $C^1(X\otimes k^u, Y\otimes k^v) = C^1(X,Y)\otimes \mathrm{Hom}_k(k^u,k^v)$ and also $U(X\otimes k^u, Y\otimes k^v) = U(X,Y) \otimes \mathrm{Hom}_k(k^u,k^v)$. Therefore, from Lemma 5.1 and from the commutativity of the diagram

$$\begin{array}{ccc} C^0(X\otimes k^u, Y\otimes k^v) & \xrightarrow{\delta_{X\otimes k^u, Y\otimes k^v}} & U(X\otimes k^u, Y\otimes k^v) \\ \| & & \| \\ C^0(X,Y)\otimes \mathrm{Hom}_k(k^u,k^v) & \xrightarrow{\delta_{X,Y}\otimes 1_{\mathrm{Hom}(k^u,k^v)}} & U(X,Y)\otimes \mathrm{Hom}_k(k^u,k^v) \end{array}$$

we obtain $\mathrm{Ext}^1_\Lambda(X\otimes k^u, Y\otimes k^v)\cong \mathrm{Ext}^1_\Lambda(X,Y)\otimes \mathrm{Hom}_k(k^u,k^v)$.

Let $\phi_1,\ldots,\phi_d$ be a basis of $U(X,Y)$. Then $\phi_1+\mathrm{Im}(\delta_{X,Y}),\ldots,\phi_n+\mathrm{Im}(\delta_{X,Y})$ form a basis of $\mathrm{Ext}^1_\Lambda(X,Y)$. Now any element in $\mathrm{Ext}^1_\Lambda(X^{\oplus u},Y^{\oplus v})$ is given by an expression $\sum_{m=1}^n (f_\alpha^{(m)}\otimes A_m)$, where $A_m\in \mathrm{Hom}_k(k^u,k^v)$ and $\phi_m=[f_\alpha^{(m)}]_{\alpha\in Q_1}$. Therefore an exceptional Λ-module M that appears in the sequence $0\to Y^{\oplus v}\to M\to X^{\oplus u}\to 0$ has the form

$$M=\left(Y_{\vec{x}}^{\oplus v}\oplus X_{\vec{x}}^{\oplus u}, \left[\begin{array}{c|c} Y_\alpha^{\oplus v} & \varphi_\alpha \\ \hline 0 & X_\alpha^{\oplus u}\end{array}\right]\right)_{0\le\vec{x}\le\vec{c},\,\alpha\in Q_1}, \qquad \varphi_\alpha=\sum_{m=1}^n (f_\alpha^{(m)}\otimes A_m),$$

for an exceptional $\Theta(n)$-representation $k^v \underset{A_n}{\overset{A_1}{\leftleftarrows\vdots}} k^u$ of the generalized Kronecker algebra. An explicit basis for $U(X,Y)$ will be constructed in the next section.

Now we will focus on exceptional modules over the generalized Kronecker algebra. The exceptional modules in this case are known; they are preprojective or preinjective and can be represented by $0,1$ matrices [Rin88]. For recent results concerning modules over t generalized Kronecker algebra we refer to [Rin13], [Rin18], [Rin16], [Wei10].

LEMMA 5.2. *Let* $V=\; k^v \underset{A_n}{\overset{A_1}{\leftleftarrows\vdots}} k^u$ *be an exceptional representation of the quiver* $\Theta(n)$ *and let* $A_m=[a_{i,j}^{(m)}]$ *for* $m=1,\ldots,n$. *Then for each pair* (i,j) *of natural numbers there is at most one* m *such* $a_{i,j}^{(m)}\neq 0$.

Proof. We will use the description of the extension space to show that if for two matrices A_1 and A_2 of an exceptional representation V of $\Theta(n)$ a non-zero entry appears in the same row and column, then $\mathrm{Ext}^1_{k\Theta(n)}(V,V)\neq 0$. Consider the map $\delta=\delta_{V,V}: C^0(V,V)\to C^1(V,V)$, where

$$C^0(V,V)=\mathrm{Hom}_k(k^v,k^v)\oplus \mathrm{Hom}_k(k^u,k^u),$$

$$C^1(V,V)=\bigoplus_{m=1}^n \mathrm{Hom}_k(k^u,k^v).$$

Then for $(f,g) \in C^0(V,V)$ we have

$$\delta(f,g) = \bigoplus_{m=1}^{n} (fA_m - A_m g).$$

The vector space $C^0(V,V)$ has a basis of the form $(e^v_{i,j}, 0)$ for $1 \le i,j \le v$, and $(0, e^u_{i,j})$ for $1 \le i,j \le u$, where $e^*_{i,j}$ is an elementary matrix with one non-zero element (equal to 1) in the $i \times j$-place. Because $A_m = [a^{(m)}_{i,j}]$ for $m = 1, \dots, n$, $\mathrm{Im}(\delta)$ is generated by the elements

$$\delta(e^v_{i,j}, 0) = \bigoplus_{m=1}^{n} e^v_{i,j} A_m = \bigoplus_{m=1}^{n} \begin{bmatrix} 0 & \cdots & 0 & a^{(m)}_{1,j} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a^{(m)}_{v,j} & 0 & \cdots & 0 \end{bmatrix},$$

$$\delta(0, e^u_{i,j}) = \bigoplus_{m=1}^{n} A_m e^u_{i,j} = \bigoplus_{m=1}^{n} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ a^{(m)}_{i,1} & a^{(m)}_{i,2} & \cdots & a^{(m)}_{i,u} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Without loss of generality we can assume that $a^{(m)}_{1,1} \neq 0$ for $m = 1, 2$. Then $x = e_{1,1} \oplus 0 \oplus \cdots \oplus 0 \in C^1(V,V)$ and $x \notin \mathrm{Im}(\delta)$. Therefore

$$\mathrm{Ext}^1_{k\Theta(n)}(V,V) \cong C^1(V,V)/\mathrm{Im}(\delta) \neq 0. \ \blacksquare$$

6. A construction of a basis for $U(X,Y)$. Let Λ be a canonical algebra of the type $\underline{p} = (p_1, \dots, p_t)$ with parameters $\underline{\lambda} = (\lambda_2 = 0, \lambda_3 = 1, \dots, \lambda_t)$. A representation $M = (\{M_{\vec{x}}\}_{\vec{x} \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1})$ of an exceptional Λ-module with positive rank is called *acceptable* if it satisfies the following conditions:

C1. $M_{\alpha_1^{(1)}}, M_{\alpha_1^{(3)}}, M_{\alpha_1^{(4)}}, \dots, M_{\alpha_1^{(t)}}$ have all entries of the form $\lambda_a - \lambda_b$ for some $a, b \ge 2$.
C2. All other matrices M_α have all entries 0 or 1.
C3. For each path $\omega^{(2)}_{u,v} : (u-1)\vec{x}_2 \to v\vec{x}_2$ all entries of $M_{\omega^{(2)}_{u,v}}$ are 0 or 1.
C4. For each path $\omega^{(i)}_{1,v} : 0 \to v\vec{x}_i$, where $i \neq 2$, all entries of $M_{\omega^{(i)}_{1,v}}$ are of the form $\lambda_a - \lambda_b$ for some $a, b \ge 2$.
C5. For each path $\omega^{(i)}_{u,v} : (u-1)\vec{x}_i \to v\vec{x}_i$ where $i \neq 2$ and $u \ge 2$, all entries of $M_{\omega^{(i)}_{u,v}}$ are 0 or 1.

The following lemma [Mel07, Lemma 3.4] is useful.

LEMMA 6.1. *Let M be an acceptable representation of a module in* $\mathrm{mod}_+(\Lambda)$. *Then by base change we can assume that*

$$M_{\alpha_j^{(i)}} = \left[\begin{array}{ccc} & I_n & \\ \hline 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{array}\right] \qquad \text{for } 2 \leq j \leq p_i,\ i = 3, \ldots, t.$$

In addition, all entries of $M_{\alpha_1^{(i)}}$ are of the form $\lambda_a - \lambda_b$ for some $a, b \geq 2$. ■

For an exceptional pair (X, Y) with acceptable representations of X and Y we will construct a basis of $U(X, Y)$ for which each basis vector has all entries of the form $\lambda_a - \lambda_b$. For $\mathrm{rk}\, X, \mathrm{rk}\, Y > 0$ this was done in [Mel07].

LEMMA 6.2. *Let X and Y be Λ-modules in* $\mathrm{mod}_+(\Lambda)$ *with acceptable representations. Then there is a basis $F^{(1)}, \ldots, F^{(d)}$ of $U(X, Y)$, where $F^{(j)} = [f_\alpha^{(j)}]_{\alpha \in Q_1}$; which has the following properties:*

(i) *All entries of the matrix $f_{\alpha_j^{(2)}}$ for $1 \leq j \leq p_i$ are 0 or 1.*
(ii) *All entries of $f_{\alpha_j^{(i)}}$ for $2 \leq j \leq p_i$ and $i = 1, 3, 4, \ldots, t$ are 0 or 1.*
(iii) *All entries of $f_{\alpha_1^{(i)}}$ for $i = 1, 3, 4, \ldots, t$ are of the form $\lambda_a - \lambda_b$.*

Note, that in the sequence $(\star)$ of Schofield induction the Λ-module Y always has positive rank, but X can have rank 0. In this situation, we need one more lemma.

LEMMA 6.3. *Let Y and X be exceptional Λ-modules such that $Y \in \mathrm{mod}_+(\Lambda)$ and $X \in \mathrm{mod}_0(\Lambda)$. Assume that X and Y have acceptable representations and X lies in the exceptional tube corresponding to the ith arm of the canonical algebra. Then there is a basis $F^{(1)}, \ldots, F^{(d)}$ of $U(X, Y)$, where $F^{(j)} = [f_\alpha^{(j)}]_{\alpha \in Q_1}$, which has the following properties:*

(i) *All entries of $f_{\alpha_j^{(2)}}$ for $1 \leq j \leq p_2$ are 0 or 1.*
(ii) *All entries of $f_{\alpha_j^{(i)}}$ for $1 \leq j \leq p_i$ are 0 or 1.*
(iii) *All entries of $f_{\alpha_1^{(m)}}$ for $m = 1, 3, 4, \ldots, t$ with $m \neq i$ are of the form $\lambda_a - \lambda_b$.*
(iv) *All entries of the matrix $f_{\alpha_j^{(m)}}$ for $2 \leq j \leq p_m$ and $m = 3, 4, \ldots, t$ are 0 or 1.*

Proof. Since X lies in the exceptional tube corresponding to the ith arm of the canonical algebra, it has a representation of the form $S_a^{[l]}$ from Section 3, such that either

(1) $1 \leq a < p_i$ and $0 < l < p_i - a$,
(2) $1 \leq a < p_i$ and $p_i - a < l < p_i$, or
(3) $a = p_i$ and $0 < l < p_i$.

In particular, all vector spaces of $S_a^{[l]}$ are zero- or one-dimensional.

CASE (1). From the shape of $S_a^{[l]}$ any element of the subspace $U(X,Y)$ has the form

$$F = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & f_{\alpha_a^{(i)}} & \cdots & f_{\alpha_{a+l-1}^{(i)}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

In addition, the condition describing the subspace $U(X,Y)$ is always satisfied.

Now we fix j such that $a \leq j < a + l$. Let e_r denote a matrix unit (a matrix with one entry 1 in row r, and the remaining entries 0). Then e_r is in

$$\mathrm{Hom}_k(X_{j\vec{x}_i}, Y_{(j-1)\vec{x}_i}) = \mathrm{Hom}_k(k, Y_{(j-1)\vec{x}_i}),$$

where $1 \leq r \leq \dim_k Y_{(j-1)\vec{x}_i}$ and

$$F_j^r = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & e_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

belongs to $U(X,Y)$ (e_r lies in the jth column). It is easy to check that the F_j^r for $1 \leq r \leq \dim_k Y_{(j-1)\vec{x}_i}$ and $a \leq j < a + l$ form a basis of $U(X,Y)$.

CASE (2). Any element of $U(X,Y)$ has the form

$$F_j^r = \begin{bmatrix} f_{\alpha_1^{(1)}} & \cdots & f_{\alpha_{s-1}^{(1)}} & f_{\alpha_s^{(1)}} & \cdots & f_{\alpha_{a-1}^{(1)}} & f_{\alpha_a^{(1)}} & \cdots & f_{\alpha_{p_1}^{(1)}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & & \\ f_{\alpha_1^{(i-1)}} & \cdots & f_{\alpha_{s-1}^{(i-1)}} & f_{\alpha_s^{(i-1)}} & \cdots & f_{\alpha_{a-1}^{(i-1)}} & f_{\alpha_a^{(i-1)}} & \cdots & f_{\alpha_{p_{i-1}}^{(1)}} \\ f_{\alpha_1^{(i)}} & \cdots & f_{\alpha_{s-1}^{(i)}} & 0 & \cdots & 0 & f_{\alpha_a^{(i)}} & \cdots & f_{\alpha_{p_i}^{(i)}} \\ f_{\alpha_1^{(i+1)}} & \cdots & f_{\alpha_{s-1}^{(i+1)}} & f_{\alpha_s^{(i+1)}} & \cdots & f_{\alpha_{a-1}^{(i+1)}} & f_{\alpha_a^{(i+1)}} & \cdots & f_{\alpha_{p_{i+1}}^{(1)}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & & \\ f_{\alpha_1^{(t)}} & \cdots & f_{\alpha_{s-1}^{(t)}} & f_{\alpha_s^{(t)}} & \cdots & f_{\alpha_{a-1}^{(t)}} & f_{\alpha_a^{(t)}} & \cdots & f_{\alpha_{p_t}^{(t)}} \end{bmatrix},$$

where the condition describing $U(X,Y)$ has the following shape:

$$\begin{aligned} &Y_{\omega_{1,p_i-1}^{(i)}} f_{\alpha_{p_i}^{(i)}} + \cdots + Y_{\omega_{1,a-1}^{(i)}} f_{\alpha_a^{(i)}} \\ &\qquad = Y_{\omega_{1,p_1-1}^{(1)}} f_{\alpha_{p_1}^{(1)}} + Y_{\omega_{1,p_1-2}^{(1)}} f_{\alpha_{p_1-1}^{(1)}} + \cdots + Y_{\alpha_1^{(1)}} f_{\alpha_2^{(1)}} + f_{\alpha_1^{(1)}} \\ &\qquad\quad + \lambda_i\big(Y_{\omega_{1,p_2-1}^{(2)}} f_{\alpha_{p_2}^{(2)}} + Y_{\omega_{1,p_2-2}^{(2)}} f_{\alpha_{p_2-1}^{(2)}} + \cdots + Y_{\alpha_1^{(2)}} f_{\alpha_2^{(2)}} + f_{\alpha_1^{(2)}}\big) \end{aligned}$$

and for $j \in \{3,4,\dots,t\}$ and $j \neq i$ we get

$$\begin{aligned} &Y_{\omega_{1,p_j-1}^{(j)}} f_{\alpha_{p_j}^{(j)}} + Y_{\omega_{1,p_j-2}^{(j)}} f_{\alpha_{p_j-1}^{(j)}} + \cdots + Y_{\alpha_1^{(j)}} f_{\alpha_2^{(j)}} + f_{\alpha_1^{(j)}} \\ &\qquad = Y_{\omega_{1,p_1-1}^{(1)}} f_{\alpha_{p_1}^{(1)}} + Y_{\omega_{1,p_1-2}^{(1)}} f_{\alpha_{p_1-1}^{(1)}} + \cdots + Y_{\alpha_1^{(1)}} f_{\alpha_2^{(1)}} + f_{\alpha_1^{(1)}} \\ &\qquad\quad + \lambda_i\big(Y_{\omega_{1,p_2-1}^{(2)}} f_{\alpha_{p_2}^{(2)}} + Y_{\omega_{1,p_2-2}^{(2)}} f_{\alpha_{p_2-1}^{(2)}} + \cdots + Y_{\alpha_1^{(2)}} f_{\alpha_2^{(2)}} + f_{\alpha_1^{(2)}}\big). \end{aligned}$$

We fix j such that $2 \leq j \leq p_i$. Again e_r is a matrix unit that belongs to

$$\mathrm{Hom}_k(X_{j\vec{x}_1}, Y_{(j-1)\vec{x}_1}) = \mathrm{Hom}_k(k, Y_{(j-1)\vec{x}_1}) \quad \text{for } 1 \leq r \leq \dim_k Y_{(j-1)\vec{x}_1}.$$

Then the element

$$F^r_{\alpha_j^{(1)}} = \begin{bmatrix} -Y_{\omega_{1,j-1}^{(1)}} e_r & 0 & \cdots & 0 & e_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

(where e_r lies in the jth column) belongs to $U(X,Y)$.

We fix j such that $1 \leq j \leq p_2$ and let e_r belong to

$$\mathrm{Hom}_k(X_{j\vec{x}_2}, Y_{(j-1)\vec{x}_2}) = \mathrm{Hom}_k(k, Y_{(j-1)\vec{x}_2}) \quad \text{for } 1 \leq r \leq \dim_k Y_{(j-1)\vec{x}_2}.$$

Then the element

$$F^r_{\alpha^{(2)}_j} = \begin{bmatrix} -\lambda_i Y_{\omega^{(2)}_{1,j-1}} e_r & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & e_r & 0 & \cdots & 0 \\ (\lambda_3 - \lambda_i) Y_{\omega^{(2)}_{1,j-1}} e_r & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (\lambda_{i-1} - \lambda_i) Y_{\omega^{(2)}_{1,j-1}} e_r & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ (\lambda_{i+1} - \lambda_i) Y_{\omega^{(2)}_{1,j-1}} e_r & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (\lambda_t - \lambda_i) Y_{\omega^{(2)}_{1,j-1}} e_r & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

belongs to $U(X, Y)$.

Next, assume that $3 \le m \le t$, $m \ne i$ and $1 < j \le p_m$. Let e_r be a matrix unit in

$$\mathrm{Hom}_k(X_{j\vec{x}_m}, Y_{(j-1)\vec{x}_m}) = \mathrm{Hom}_k(k, Y_{(j-1)\vec{x}_m}) \quad \text{for } 1 \le r \le \dim_k Y_{(j-1)\vec{x}_k}.$$

Then the element

$$F^r_{\alpha^{(m)}_j} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -Y_{\omega^{(m)}_{1,j-1}} e_r & 0 & \cdots & 0 & e_r & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

belongs to $U(X, Y)$.

Now, we fix j such that $j \in \{a, a+1, \ldots, p_i\}$ and let e_r be a matrix unit in

$$\mathrm{Hom}_k(X_{j\vec{x}_i}, Y_{(j-1)\vec{x}_i}) = \mathrm{Hom}_k(k, Y_{(j-1)\vec{x}_i}),$$

where $1 \le r \le \dim_k Y_{(j-1)\vec{x}_i}$. Then the element

$$F^r_{\alpha^{(i)}_j} = \begin{bmatrix} Y_{\omega^{(i)}_{1,j-1}} e_r & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ Y_{\omega^{(i)}_{1,j-1}} e_r & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{\omega^{(i)}_{1,j-1}} e_r & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & e_r & 0 & \cdots & 0 \\ Y_{\omega^{(i)}_{1,j-1}} e_r & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{\omega^{(i)}_{1,j-1}} e_r & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

belongs to $U(X,Y)$.

Let $j \in \{1,\dots,s-1\}$ and let e_r belong to $\mathrm{Hom}_k(X_{j\vec{x}_i}, Y_{(j-1)\vec{x}_i}) = \mathrm{Hom}_k(k, Y_{(j-1)\vec{x}_i})$ for $1 \le r \le \dim_k Y_{(j-1)\vec{x}_i}$. Then the element

$$F^r_{\alpha^{(i)}_j} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & e_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

belongs to $U(X,Y)$, where e_r lies in the jth column and ith row.

It is easy to check that the F^r_α form a basis of $U(X,Y)$. Finally, we must check that the matrices f^r_α of the basis vectors F^r_α have the desired entries. Because the representation for Y is acceptable, the matrices $Y_{\omega^{(m)}_{u,v}}$ have only entries 0, ± 1, $\pm\lambda_a$, $\lambda_a - \lambda_b$. Hence $Y_{\omega^{(m)}_{1,j-1}} e_r$ has the same entries. In addition, for $m = 2$, all entries of $Y_{\omega^{(2)}_{u,v}}$ are 0 or 1. Therefore the matrices $(\lambda_a - \lambda_b) Y_{\omega^{(2)}_{1,j-1}} e_r$ have only entries 0, ± 1, $\pm\lambda_a$, $\lambda_a - \lambda_b$.

Case (3) is similar to case (2). ∎

Note that entries of the form $\lambda_a - \lambda_b$ only occur in a basis vector of $U(X,Y)$ if they appear in the acceptable representations of X or Y. In particular if X and Y are modules of rank 1, then by Proposition 3.1 all basis vectors of $U(X,Y)$ have only entries 0, ± 1 and $\pm\lambda_i$.

7. Proof of the main theorem

PROPOSITION 7.1 (Induction step). *Let M be an exceptional module over a canonical algebra Λ with* $\operatorname{rk} M \geq 2$. *Let (X, Y) be an orthogonal exceptional pair of Λ-modules, obtained by applying Schofield induction to M. If X and Y allow acceptable representations, then also M allows an acceptable representation.*

Proof. We will use the basis $F^{(1)} = [f_\alpha^{(1)}]_{\alpha\in Q_1}, \dots, F^{(n)} = [f_\alpha^{(n)}]_{\alpha\in Q_1}$ of the subspace $U(X,Y)$ from Lemma 6.2 or Lemma 6.3. Since M belongs to $\mathcal{F}(X,Y)$, it has the form

$$M = \left(Y_{\vec{x}}^{\oplus v} \oplus X_{\vec{x}}^{\oplus u}, \left[\begin{array}{c|c} Y_\alpha^{\oplus v} & \varphi_\alpha \\ \hline 0 & X_\alpha^{\oplus u} \end{array} \right] \right)_{0\leq \vec{x}\leq \vec{c},\, \alpha\in Q_1}, \qquad \varphi_\alpha = \sum_{m=1}^{n} (f_\alpha^{(m)} \otimes A_m),$$

for an exceptional $\Theta(n)$-representation $k^v \underset{A_n}{\overset{A_1}{\longleftarrow \vdots \longleftarrow}} k^u$. Recall that all matrices $A_1, \dots, A_n$ have all entries 0 or 1 (see [Rin88]) and moreover from Lemma 5.2 non-zero coefficients in consecutive matrices $A_1, \dots, A_n$ occur in different places. Therefore the matrix $\sum_{m=1}^n (f_\alpha^{(m)} \otimes A_m)$ has the same entries as the matrices of the basis vectors $F^{(1)}, \dots, F^{(n)}$ of $U(X,Y)$. Therefore, since X and Y are acceptable, the matrix $M_{\alpha_1^{(i)}}$ has all entries of the form $\lambda_a - \lambda_b$ for $i = 1, 3, 4, \dots, t$, and for $i = 2$ only 0 and 1 appear. Next, $M_{\alpha_j^{(i)}}$ is a $0,1$ matrix for $2 \leq j \leq p_i$ and $i = 1, \dots, t$.

Now, we must check that for each path $\omega_{l,m}^{(i)} = \alpha_m^{(i)} \dots \alpha_l^{(i)}$ the matrix $M_{\omega_{l,m}^{(i)}} = M_{\alpha_l^{(i)}} \dots M_{\alpha_m^{(i)}}$ has expected entries. After standard calculations we obtain

$$M_{\omega_{l,m}^{(i)}} = \left[\begin{array}{c|c} \big(Y_{\omega_{l,m}^{(i)}}\big)^{\oplus v} & \sum Y f X_{\omega_{l,m}^{(i)}} \\ \hline 0 & \big(X_{\omega_{l,m}^{(i)}}\big)^{\oplus u} \end{array} \right],$$

where $\sum YfX_{\omega_{l,m}^{(i)}}$ denotes

$$\sum_{j=1}^{n} \big\{ Y_{\omega_{l,m-1}^{(i)}} f_{\alpha_m^{(i)}}^{(j)} + Y_{\omega_{l,m-2}^{(i)}} f_{\alpha_{m-1}^{(i)}}^{(j)} X_{\alpha_m^{(i)}} + \dots + f_{\alpha_l^{(i)}}^{(j)} X_{\omega_{l+1,m}^{(i)}} \big\} \otimes A_j.$$

Since X and Y are acceptable, $Y_{\omega_{l,m}^{(i)}}$ and $X_{\omega_{l,m}^{(i)}}$ have desired entries. Again $\sum YfX_{\omega_{l,m}^{(i)}}$ has the same entries as

$$Y_{\omega_{l,m-1}^{(i)}} f_{\alpha_m^{(i)}}^{(j)} + Y_{\omega_{l,m-2}^{(i)}} f_{\alpha_{m-1}^{(i)}}^{(j)} X_{\alpha_m^{(i)}} + \dots + f_{\alpha_l^{(i)}}^{(j)} X_{\omega_{l+1,m}^{(i)}}.$$

Now the statement concerning the entries of $M_{\omega_{l,m}^{(i)}}$ follows from the explicit description of $F^{(i)}$ by a case by case inspection. ■

Note that entries of the form $\lambda_a - \lambda_b$ appear only for regular modules. This means that if in the tree (4.1) there are only vector bundles, then each module in this tree (after translations) can be realized by matrices with entries $0, \pm 1, \pm\lambda_a$.

Proof of the Main Theorem. We use induction on the rank of the exceptional module. The description of exceptional modules of rank 0 and 1 in Section 3 gives the base case of induction. Let M be an exceptional Λ-module of rank r and assume that $r \geq 2$. Then M corresponds to an exceptional vector bundle over the weighted projective line $\mathbb{X}$ associated to Λ. By repeated use of Schofield induction, we obtain the tree (4.1) in the category $\mathrm{coh}(\mathbb{X})$ for M. Then from Corollary 4.2 we can shift all sheaves in this tree so that each of them becomes a Λ-module. Therefore, we can assume that "almost all" sheaves in (4.1) belong to $\mathrm{mod}(\Lambda)$. Since all components in this tree have smaller rank than M, they have acceptable representations. Therefore the claim follows from Proposition 7.1. ■

Acknowledgments. The result is part of the PhD thesis of the first author at Szczecin University in 2017. The authors are grateful to C. M. Ringel for a helpful discussion concerning the paper [Rin88].

REFERENCES

[AHR92] D. M. Arnold, R. Hunter and F. Richman, *Abelian groups and representations of finite partially ordered sets*, Forum Math. 4 (1992), 349–357.

[ASS06] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, Cambridge Univ. Press, 2006.

[Bäc72] K. J. Bäckström, *Orders with infinitely many indecomposable lattices*, PhD thesis, Göteborg, 1972.

[Bir34] G. Birkhoff, *Subgroups of Abelian groups*, Proc. London Math. Soc. 38 (1934), 385–401.

[Bon81] K. Bongartz, *Tilted algebras*, in: M. Auslander and E. Lluis (eds.), Representations of Algebras (Puebla, 1980), Lecture Notes in Math. 903, Springer, Berlin, 1981, 26–38.

[DM19] P. Dowbor and H. Meltzer, *Exceptional objects of nilpotent operators of degree 3 with two incomparable subspaces*, preprint, Toruń, 2018.

[DMM10] P. Dowbor, H. Meltzer and A. Mróz, *An algorithm for the construction of exceptional modules over tubular canonical algebras*, J. Algebra 323 (2010), 2710–2734.

[DMM14a] P. Dowbor, H. Meltzer and A. Mróz, *Parametrizations for integral slope homogeneous modules over tubular canonical algebras*, Algebras Represent. Theory 17 (2014), 321–356.

[DMM14b] P. Dowbor, H. Meltzer and A. Mróz, *An algorithm for the construction of parametrizing bimodules for homogeneous modules over tubular canonical algebras*, Algebras Represent. Theory 17 (2014), 357–405.

[DMS19] P. Dowbor, H. Meltzer and M. Schmidmeier, *The "0,1" property of exceptional objects for nilpotent operators of degree 6 with one invariant subspace*, J. Pure Appl. Algebra 223 (2019), 3150–3203.

[Drä01] P. Dräxler, *Normal forms for representations of representation-finite algebras*, J. Symbolic Comput. 32 (2001), 491–497.

[Gab71] P. Gabriel, *Unzerlegbare Darstellungen I*, Manuscripta Math. 6 (1971), 71–103.

[GL87] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*, in: Singularities, Representations of Algebras, and Vector Bundles, Lecture Notes in Math. 1273, Springer, 1987, 265–297.

[GKM12] M. Grzecza, S. Kasjan and A. Mróz, *Tree matrices and a matrix reduction algorithm of Belitskii*, Fund. Inform. 118 (2012), 253–279.

[KęM11] D. E. Kędzierski and H. Meltzer, *Indecomposable representations for extended Dynkin quivers of type* $\widetilde{E}_8$, Colloq. Math. 124 (2011), 95–116.

[KęM13] D. E. Kędzierski and H. Meltzer, *Schofield induction for sheaves on weighted projective lines*, Comm. Algebra 41 (2013), 2033–2039.

[Kle72] M. M. Kleiner, *Partially ordered sets of finite type*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklova 28 (1972), 32–41 (in Russian); English transl.: J. Soviet Math. 3 (1975), 607–615.

[KoM08] S. Komoda and H. Meltzer, *Indecomposable modules for domestic canonical algebras in arbitrary characteristic*, Int. J. Algebra 2 (2008), 153–161.

[KuM07a] D. Kussin and H. Meltzer, *Indecomposable modules for domestic canonical algebras*, J. Pure Appl. Algebra 211 (2007), 471–483.

[KuM07b] D. Kussin and H. Meltzer, *Indecomposable representations for extended Dynkin quivers*, arXiv:math/0612453 (2007).

[KLM13a] D. Kussin, H. Lenzing and H. Meltzer, *Nilpotent operators and weighted projective lines*, J. Reine Angew. Math. 685 (2013), 33–71.

[KLM13b] D. Kussin, H. Lenzing and H. Meltzer, *Triangle singularities, ADE chains and weighted projective lines*, Adv. Math. 237 (2013), 194–251.

[KLM18] D. Kussin, H. Lenzing and H. Meltzer, *Nilpotent operators and flags of invariant subspaces*, preprint, Paderborn, 2018.

[LeM93] H. Lenzing and H. Meltzer, *Sheaves on a weighted projective line of genus one, and representations of a tubular algebra*, in: Representations of Algebras (Ottawa, ON, 1992), CMS Conf. Proc. 14, Amer. Math. Soc., Providence, RI, 1993, 313–337.

[LdP97] H. Lenzing and J. A. de la Peña, *Wild canonical algebras*, Math. Z. 224 (1997), 403–425.

[Mel07] H. Meltzer, *Exceptional modules for tubular canonical algebras*, Algebras Represent. Theory 10 (2007), 255–267.

[Rin76] C. M. Ringel, *Representations of K-species and bimodules*, J. Algebra 41 (1976), 269–302.

[Rin84] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.

[Rin88] C. M. Ringel, *Exceptional modules are tree modules*, Linear Algebra Appl. 275-276 (1998), 471–493.

[Rin13] C. M. Ringel, *Indecomposable representations of the Kronecker quivers*, Proc. Amer. Math. Soc. 141 (2013), 115–121.

[Rin16] C. M. Ringel, *The elementary 3-Kronecker modules*, preprint, Bielefeld, 2016.

[Rin18] C. M. Ringel, *Kronecker modules generated by modules of length 2*, in: Representations of Algebras, Contemp. Math. 705, Amer. Math. Soc., 2018, 189–214.

[RSc08] C. M. Ringel and M. Schmidmeier, *Invariant subspaces of nilpotent linear operators, I*, J. Reine Angew. Math. 614 (2018), 1–52.

[Sch91] A. Schofield, *Semi-invariants of quivers*, J. London Math. Soc. 43 (1991), 385–395.

[Sim91] D. Simson, *On representations of partially ordered sets of tame type*, Universität Bielefeld, Sonderforschungsbereich 343, “Diskrete Strukturen in der Mathematik”, Ergänzungsreihe 91–004, 1991, 1–101.

[Sim18] D. Simson, *Representation-finite Birkhoff type problems for nilpotent linear operators*, J. Pure Appl. Algebra 222 (2018), 2181–2198.

[SSk06] D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras. Volume 3: Representation-Infinite Tilted Algebras*, Cambridge Univ. Press, 2006.

[Wei10] T. Weist, *Tree modules of the generalised Kronecker quiver*, J. Algebra 323 (2010), 1107–1138.

Dawid Edmund Kędzierski, Hagen Meltzer
Institute of Mathematics
Szczecin University
70-451 Szczecin, Poland
E-mail: dawid.kedzierski@usz.edu.pl
hagen.meltzer@usz.edu.pl